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Synchronization induced by Langevin dynamics

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Equilibrium Langevin dynamics of one-dimensional Lennard-Jones chains is studied. It is shown that depending on the noise strength, the friction constant and the number of particles, chains can synchronize, break, or remain desynchronized. Generally the synchronization time and the maximal Lyapunov exponent are found to depend on the number of particles and the ratio of noise strength to friction constant.

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Dynamics of nonlinear systems exhibiting chaotic motions shows an extreme sensitivity to even minor perturbations. This feature, also often referred to as Lorenz *''butterfly effect,''* makes long-time evolution of the systems unpredictable. Consequently, trajectories of identical chaotic systems that start their evolution from different initial points in phase space are not expected to synchronize, and using a random resetting signal should make them even ''more random.'' References [1–4] were the first to discuss very intriguing possibilities, counterintuitive to the above expectations. For example, Fahy and Hamman [4] studied a system of two noninteracting particles moving chaotically inside a confining potential. Velocities of the particles were randomly reset according to the Maxwellian distribution at regular time intervals τ. They observed that for τ less than a threshold value the distance between the trajectories was convergent exponentially to zero with increasing time, independent of the initial conditions of the particles.

Since then investigation of synchronization process in chaotic systems has been an active line of research [5–13]. Essentially, the following types of synchronization are identified: (i) synchronization through a coupling of identical chaotic systems in a drive-response manner as proposed in Ref. [3]; (ii) synchronization of identical chaotic systems by common noises (see, e.g., Refs. [4–10]); (iii) synchronization of time-delayed systems (see, e.g., Ref. [11]); (iv) partial synchronization (see, e.g., Ref. [12]) and (v) generalized synchronization between nonidentical systems [13].

The main purpose of the present Rapid Communication is to study the chaotic synchronization as induced by common external noises. For many years the subject has been quite controversial (see, e.g., [8] and references therein) and only recently it has been approached in a systematic way by Lai and Zhou [9], and by Rim *et al.* [10]. In particular, a suggestion has been put forward in Ref. [10] that any deterministic, chaotic system should synchronize given that an appropriate noise profile is applied. As a support of this conjecture a computer analysis of two logistic maps, coupled by uniform additive noise of width b and bias a, has been carried out [10]. More specifically, it was shown that in the (a,b) plane there is a line separating chaotic noisy behavior from the nonchaotic noisy one, and that the region of coalescence is characterized by the nonvanishing mean of the noise. Interestingly, on approaching the boundary, an intermittent behavior of the distance function was observed. Also, it was found that the laminar distribution as function of the laminar length follows the $-3/2$ power law, which characterizes the *on-off* intermittency. Analogous observations have been reported for the Lorenz model [6], but no analysis of the system behavior in the vicinity of the boundary was carried out.

One of the objectives of the present studies is to check the hypothesis of Rim *et al.* [10] for models with more than one degree of freedom. We are interested in realistic situations where trajectories are generated by ordinary, equilibrium Langevin dynamics, with both dissipation and noise being present. That is, we assume the noise and the friction to follow restrictions imposed by the fluctuation-dissipation theorem. Clearly, the noise of vanishing mean should be considered, in which case the synchronization is harder to get [9]. We are also interested in the system size dependence of the synchronization. A numerical evidence showing that such *''Langevin synchronization''* is indeed possible has been reported by Cattuto and Marchesoni [14] for a perturbed, stochastic, sine-Gordon equation. But no explanation as of the mechanism responsible for it has been proposed.

Interestingly, as we show by detailed studies of a one-dimensional model, the equilibrium Langevin dynamics indeed could lead to a collapse of the trajectories. It is observed for certain values of ratio of the noise strength to the friction constant, for practically all nonzero temperatures. Furthermore, the numerical results indicate that the number of particles and details of the noise profile are of secondary importance for the collapse to occur. Also, particles do not need to move in a confining potential, as in the case discussed by Fahy and Hamman [4].

In an attempt to address the above issues, we have performed computer simulations on one-dimensional chains. We used the simplest version of Langevin dynamics, where the motion of an ensemble of identical particles is described by the phenomenological Langevin equation consisting of inertial terms, force field, frictional drag, and noise, respectively. For a one-dimensional case it reads

$$m\ddot{r}_i = -\nabla_{r_i} V - m\gamma \dot{r}_i + F_i(t). \tag{1}$$

Here r_i, $i=1,\ldots,N$, is the coordinate of the ith particle of mass m of a chain, moving in a potential V. At equilibrium the friction γ is related to the noise $F_i(t)$ through the fluctuation-dissipation theorem.

The noise term, Eq. (1), describing the interaction with a thermal reservoir at temperature T, usually is represented by the Gaussian stationary random force such that

$$\overline{F_i(t)}=0, \quad \overline{F_i(t)F_j(t')}=2\delta_{ij}\delta(t-t')\gamma m k_B T, \tag{2}$$

where the coefficients of the second condition follow from the fluctuation-dissipation theorem. The term proportional to velocity represents the average force from the environment acting on the particle and giving rise to viscosity, or friction. $F_i(t)$ takes into account the rapidly varying part of the force, bearing in mind the very frequent individual impacts of molecules with the observed particles. For example, in polymer physics, the frictional drag and noise simulate the effect of individual solvent molecules acting on a polymer (see e.g., Refs. [15–17]). Clearly, the Langevin dynamics is relevant not only to polymer physics. It also plays an important role in the description of problems in mathematics, astrophysics, chemical physics, biology, laser physics, etc.

Now the question that arises is whether the Langevin dynamics indeed could yield synchronization. In a trivial sense the answer is positive. To see it, suppose that the dynamics, Eq. (1), with noise and friction switched off is chaotic, i.e., with the maximal Lyapunov exponent being positive. By switching the friction on and disregarding the chaotic part one forces the system to evolve to its ground state, which plays the role of an attractor. Consequently, the synchronization could easily be realized in practice. In the opposite case, where $F_i(t)$ is nonzero and $\gamma=0$, at least in the limit of weak $F_i(t)$, the Lyapunov exponent should remain positive preventing the system to have synchronizing properties. Consequently, for a fixed distribution of the noise there should exit γ^*, such that for $\gamma>\gamma^*$ the system synchronizes while otherwise it does not. Clearly, the above reasoning is generally valid and does not restrict to the one-dimensional chains.

Interestingly, the synchronization could also be achieved if we assume that the restrictions due to the fluctuation-dissipation theorem are fulfilled. This statement is not trivial anymore, and we do not have a general proof of it. However, we can demonstrate its validity by considering, e.g., the equilibrium Langevin dynamics of two identical copies of a one-dimensional chain composed of N particles of equal masses m. Within each chain particles are assumed to interact with their nearest-neighbors via a Lennard-Jones potential. Definitions we use are given in Ref. [16].

Particles of index i of both chains are next subjected to the same thermal bath $\{F_i\}$. Detailed simulations are carried out for two noise profiles: for the standard Gaussian one and for a non-Gaussian noise: $\{F_i=\sigma\eta_i\}$, where η_i are the random numbers uniformly distributed on the interval $-1\leqslant\eta_i\leqslant 1$. The value of the noise strength σ depends on the time increment Δt that one uses to numerically integrate Eq. (1). Assuming that the Brownian force is constant in the interval Δt and taking regard of the fluctuation-dissipation theorem we get for the Gaussian noise

$$\sigma=\sqrt{2\gamma m k_B T/\Delta t}, \tag{3}$$

in agreement with the Eq. (2). For the non-Gaussian case the factor ''2'' of the last equation should be replaced by ''6.'' As expected, for fixed γ and T, both noise profiles yield the calculated equilibrium averages that are indistinguishable to within the statistical error. For more details see Refs. [16,17], where the Langevin dynamics of the same Lennard-Jones chain has been used to model a fragmentation process in polymers. In our calculations we adopt the system of reduced units from these references.

For integration of the equations of motion the velocity Verlet algorithm [18] is used with the time step $0.000\,05\leqslant\Delta t/\tau_0\leqslant 0.05$ ($\tau_0=2\pi/\omega_0$, $\omega_0=12\sqrt{2\epsilon/ma^2}$ [16,17]). Two types of boundary conditions are applied. The first one, which we call a free chain (FC) simulation, assumes fixed position of centers of mass of both chains at a common origin and vanishing center of mass velocities. More specifically, after bringing both chains to equilibrium we fix their center of mass parameters as described above and these parameters are *preserved*, up to irrelevant fluctuations, by the Verlet algorithm. Clearly, the fluctuations arise because the total random force, ΣF_i, acting on the system follows Gaussian distribution according to the central limit theorem. To simplify our code that monitors the breaking of the chain we disregard these fluctuations by adding $\delta X_{CM}/N$ ($\delta V_{CM}/N$) to positions (velocities) of each particle at every time step, so that the parameters of the center of mass are left unchanged.

The second type of boundaries, referred to as restricted chain (RC) simulation, confine the systems to an interval $[0, L=Na]$ with $x_1=0$ and $x_N=L$. In both cases initial velocities of the particles are taken independently from the Maxwell distribution, separately for each chain. The variance of the distribution is set equal to $k_BT/2m\epsilon$, where k_BT/ϵ takes the same value as the one used in Eq. (3). Additionally, initial positions of the particles are chosen with uniform distribution about ground state configuration (regular $1-d$ lattice) with maximal displacement not exceeding $0.2a$. The systems are studied for the reduced temperature, k_BT/ϵ, varying between 0.001 and 0.05. N is restricted to 10, 20, 30, 50, 100, and 1000, but most of the simulations are carried out for $N=10$ and $N=100$. The synchronization is monitored by calculating relative distance, D, and relative velocity V between the chains: $D=\sqrt{1/N\Sigma_{i=1}^{N}(x_i^1-x_i^2)^2}$, and $V=\sqrt{1/N\Sigma_{i=1}^{N}(V_i^1-V_i^2)^2}$, where superscripts label the chains. x_i^I and V_i^I are the position and the velocity of a particle i of the chain I ($I=1,2$), respectively. We also determine the maximal Lyapunov exponent.

In the case of $\sigma=\gamma=0$ the reference system is characterized by the positive Lyapunov exponent and, consequently, no collapse is possible. However, the situation changes when

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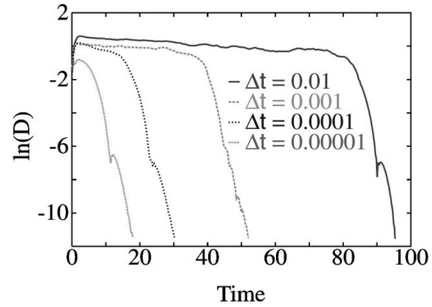

FIG. 1. Reduced time (t/τ_0) dependence of the relative distance, ln(D/a)≡ln(D), for various reduced time steps: $\Delta t/\tau_0 = 0.05$, 0.005, 0.0005, and 0.000 05, and for $k_B T/\epsilon = 0.01$ and $\gamma = 0.25\omega_0$. Both chains are composed of ten particles. The curve is erratic, but the assumed scale of the D-axis makes it look smooth. Similar behavior is found for relative velocity.

the noise and the friction are switched on. As it turns out the most intriguing are the FC simulations. In this case the force field does not confine particles within the chain and, hence, the time evolution of a single, unconfined chain for $T>0$ is such that its average dimension grows with increasing time, i.e., the chain breaks [17]. But even in this case we may observe synchronization, as illustrated in Fig. 1, although relation between synchronization, initial conditions and the noise strength still has to be elucidated. What we observe is that trajectories of two initially uncorrelated identical chains subjected to the same Langevin noise, $\{F_i\}$, may indeed collapse to a single trajectory in the sense that the average distance between them converges exponentially to zero. The collapse starts after the transient period, which is shown as plateau in Fig. 1. This plateau grows with decreasing noise strength, Eq. (3), and with increasing N.

For RC simulations the system is confined and, hence, any initial conditions can be used to start evolution. In this case we found generally that synchronization depends on the ratio of the noise strength to friction constant.

By calculating numerically the maximal Lyapunov exponent we determined the boundaries in the $(\sigma/\sigma_0,\ \gamma/\gamma_0)$ plane ($k_B T_0/\epsilon = 0.05$, $\gamma_0 = 0.25\omega_0$, $\sigma_0 = \sqrt{2mk_B T_0 \gamma_0/\Delta t}$ [16]), where the Lyapunov exponent changes sign. The results for the Gaussian noise are shown in Fig. 2 for N

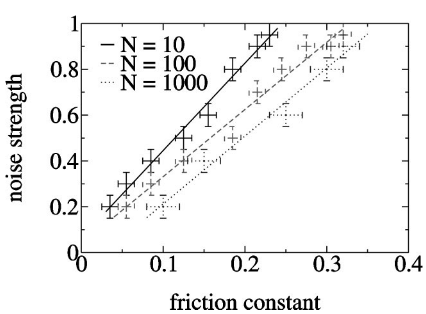

FIG. 2. Boundaries in the plane (noise strength, friction constant)$=(\sigma/\sigma_0,\ \gamma/\gamma_0)$ where the maximal Lyapunov exponent changes sign. Parameters used are $k_B T_0/\epsilon = 0.05$, $\gamma_0 = 0.25\omega_0$, $\Delta t = 0.01\tau_0$ and $\sigma_0 = \sqrt{6mk_B T_0 \gamma_0/\Delta t}$ [16]). The region below the boundary line is where the synchronization takes place.

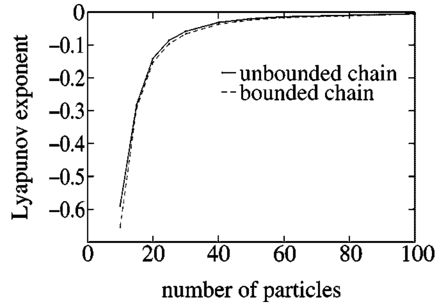

FIG. 3. Maximal Lyapunov exponent as function of the number of particles within the chain. Parameters used are $\gamma = 0.25\omega_0$, $k_B T/\epsilon = 0.05$, $\Delta t = 0.01\tau_0$, and $\sigma = \sqrt{2mk_B T\gamma/\Delta t}$ [16].

$=10$, 100, and 1000 indicating on a linear dependence between σ/σ_0 and γ/γ_0 along the boundary. Statistically the same results are obtained for the non-Gaussian noise, which we checked for $N \leqslant 100$. Owing to a very small value of the Lyapunov exponent for $N=1000$, which requires extremely long molecular dynamics runs, we have carried out detailed simulations in this case only for the Gaussian case.

The collapse occurs when the parameters $(\sigma/\sigma_0,\ \gamma/\gamma_0)$ are taken from the area below the separatrices in Fig. 2. More specifically, for fixed N and σ/σ_0 the values of γ/γ_0 should be greater than a threshold value, which (as already mentioned above) obeys the heuristic law $\sigma/\sigma_0/\gamma/\gamma_0 = \text{const}$. The corresponding temperature T/T_0 is found from condition (3), which could also be written as: $\sigma/\sigma_0 = \sqrt{\gamma T/(\gamma_0 T_0)}$, independent of the applied noise profile. Conversely, in the $(T/T_0, \gamma/\gamma_0)$-plane the separatrices are given by analogous linear relation: $(T/T_0)/(\gamma/\gamma_0) = \text{const}^2$, implying that for fixed relative temperature (T/T_0) the friction must again be greater than the threshold to observe synchronization.

A typical behavior of the Lyapunov exponent as function of N in the nonchaotic noisy region is shown in Fig. 3. For all Ns and for the model parameters studied we do not observe, within the statistical error, a crossover from nonchaotic to chaotic noisy behavior as a function of N. On approaching the boundary the distance function shows a characteristic intermittent behavior, consistent with the observations of Rim *et al.* [10]. However, owing to the large fluctuations in the transition regime, as indicated by error bars in Fig. 2, we were unable to determine unequivocally the characteristic exponent of the laminar distribution.

It is perhaps worthwhile to emphasize that the equilibrium, fixed temperature, Langevin dynamics could be carried out on both sides of the separatrices in Fig. 2; that is, either in the chaotic noisy region (positive Lyapunov exponent) or in the nonchaotic noisy one (negative Lyapunov exponent). From the discussion as given these features seem to be inherent to the structure of the Langevin equations.

In conclusion, we have investigated in detail a possibility of the chaotic synchronization in the case when the dynamics of the identical systems is governed by ordinary Langevin equations. By studying the Lennard-Jones chains of various lengths we showed that the synchronization can occur even when the frictional drag and the *symmetric* noise obey the fluctuation-dissipation theorem. This statement holds independent of the system size studied (at least up to $N=1000$). The results also seem independent of the details of the noise profile. For this nontrivial synchronization we recovered a picture suggested by Rim *et al.* [10] in which the transition from a chaotic to a nonchaotic noisy region goes through an intermittent behavior, but, owing to large fluctuations close to transition region we were unable to analyze scaling properties of the laminar distribution.

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